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Entropy formulations for a class of scalar conservations laws with space-discontinuous flux functions in a bounded domain

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Abstract In this paper, the mathematical analysis of a quasilinear parabolic–hyperbolic problem in a multidimensional bounded domain Ω is carried out. In a region Ω_p a diffusion–advection–reaction-type equation is set, while in the complementary $\Omega_h \equiv \Omega\backslash\Omega_p$, only advection–reaction terms are taken into account. First, the definition of a weak solution u is provided through an entropy inequality on the whole domain Q by using the classical Kuzhkov entropy pairs and the F. Otto framework to transcribe the boundary conditions on $\partial\Omega \cap \partial\Omega_h$. Since Γ_{hp} contains the outward characteristics for the first-order operator set in Q_h, the uniqueness proof begins by focusing on the behavior of u in the hyperbolic layer and then in the parabolic one where u fulfills a variational equality that takes into account the entered data from Q_h. The existence property uses a vanishing-viscosity method.

Keywords Conservation laws with discontinuous flux functions · Coupling of parabolic–hyperbolic equations · Entropy formulation · Entropy process solution

1 Introduction

This paper deals with the coupling of a quasilinear parabolic equation with a quasilinear hyperbolic one of first order in a multidimensional bounded domain Ω. The former is an advection–diffusion–reaction-type equation set in a region Ω_p of Ω, while the latter—set in the complementary region $\Omega_h = \Omega\backslash\Omega_p$—only contains an advection–reaction part.

As mentioned in [1], this type of problem arises from several physical applications that are modeled by a global advection–diffusion–reaction process in the whole Ω. However, in these problems, the diffusive term may be relevant only in a subregion Ω_p (which clearly depends on the problem in hand), while it can be neglected in the rest of the domain Ω, without affecting the solution appreciably.

Fluid dynamics is one of the fields that benefit greatly from a coupling approach of the type considered here. For example, we may consider viscous compressible flows around a rigid profile (e.g., an aerofoil). Physical evidence

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suggests that viscosity effects are negligible apart from a small region close to the rigid body. This means that the mathematical modeling of the problem may lead to the use of equations of different character (precisely Euler, Navier–Stokes equations) in separate regions, by dropping viscous terms when they are very small.

Another example is provided by a heat-transfer problem such as a forced incompressible flow over a heated plate. In such a case, the thermal diffusivity is much more important in the boundary layer than elsewhere (here the reduced equation of energy conservation can be assumed to describe the flow field). The velocity field can be evaluated independently from that of the temperature, while the latter is the solution to an advection–diffusion equation in which the transport field is given precisely by the (known) velocity. Away from the boundary layer, the diffusive term may be neglected.

We complete this introduction with yet another example, within the framework of infiltration processes through a stratified subsoil viewed as an heterogeneous porous medium with different geological characteristics in each layer, and such that, depending on the physical properties of the rock, the diffusivity effects may be neglected with respect to those related to transport. This approach has mainly motivated the previous studies of [2] and [3].

2 Mathematical setting

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 1$ (in practical $n = 3$), such that $\overline{\Omega} = \overline{\Omega}_h \cup \overline{\Omega}_p$; Ω_h (*hyperbolic zone*) and Ω_p (*parabolic zone*) being two disjoint bounded domains with Lipschitz boundaries denoted by $\Gamma_l = \partial\Omega_l$, $l \in \{h, p\}$ and $\Gamma_{hp} = \Gamma_h \cap \Gamma_p$. Let T be a finite positive real. We are interested in the uniqueness and existence of a measurable and bounded function u on $Q \equiv]0, T[\times\Omega$ satisfying

$$\partial_t u - \sum_{j=1}^{n} \partial_{x_j}(\mathbb{I}_{\Omega_p}(x)\partial_{x_j}\phi(u) + K_j(x, u)) + g(t, x, u) = 0 \text{ on } Q, \tag{1}$$

$$u = 0 \text{ on }]0, T[\times\partial\Omega,\ u(0, .) = u_0 \text{ on } \Omega, \tag{2}$$

with $\mathbb{I}_{\Omega_p}(x) = 1$ if x belongs to Ω_p and 0 otherwise,

$$K_j(x, u) = K_h(u)B_{h,j}(x)\mathbb{I}_{\Omega_h}(x) + K_p(u)B_{p,j}(x)\mathbb{I}_{\Omega_p}(x),\ j \in \{1, \dots, n\},$$
$$g(t, x, u) = g_p(t, x, u)\mathbb{I}_{\Omega_p}(x) + g_h(t, x, u)\mathbb{I}_{\Omega_h}(x).$$

We set $\Gamma_i = \partial\Omega_i$, $i \in \{h, p\}$, $\Gamma_{hp} = \Gamma_h \cap \Gamma_p$ so that $\mathcal{H}^{n-1}(\overline{\Gamma_{hp}} \cap (\overline{\Gamma_i \backslash \Gamma_{hp}})) = 0$, where $\mathcal{H}^q$ is the q-dimensional Hausdorff measure.

The vector fields $\mathbf{B}_i = (B_{i,1}, \dots, B_{i,n})$ are elements of $W^{2,+\infty}(\Omega_i)^n$ and such that

$$\Gamma_{hp} \subset \{\bar{\sigma} \in \Gamma_h, \mathbf{B}_h(\bar{\sigma}) \cdot \nu_h \leq 0\}. \tag{3}$$

where ν_i denotes the outward normal unit vector defined $\mathcal{H}^n$-a.e. on $]0, T[\times\Gamma_i$.

The initial datum u_0 belongs to $L^\infty(\Omega)$ and the transport term K_p is Lipschitz continuous on $\mathbb{R}$ with a constant $\mathcal{K}'_p$ and K_h is a *nondecreasing* Lipschitz continuous function on $\mathbb{R}$, with a constant $\mathcal{K}'_h$. Besides, for i in $\{h, p\}$, the reaction term g_i is measurable and bounded function on $]0, T[\times\Omega_i \times \mathbb{R}$ such that

$$\exists M'_{g_i} \in \mathbb{R},\ \text{a.e. on }]0, T[\times\Omega_i \times \mathbb{R}, |\partial_u g_i| \leq M'_{g_i}.$$

We assume that there exist $a \in \mathbb{R}^-$ and $b \in \mathbb{R}^+$ such that $a < b$ and ϕ is a nondecreasing function of $W^{1,+\infty}(a, b)$, with $\phi(0) = 0$ satisfying

$$\phi^{-1} \text{ exists on } [\phi(a), \phi(b)],$$

that is fulfilled, in particular, when $\{x \in [a, b], \phi'(x) = 0\}$ has a zero Lebesgue measure. Observe that, thanks to the Rademacher's Theorem, the function ϕ is differentiable a.e. on $[a, b]$. This will be useful to state the existence result for (1)–(2).

2.1 Notations and functional spaces

In the sequel, σ (resp. $\bar{\sigma}$) is a variable of $\Sigma_i \equiv]0, T[\times\Gamma_i$ (resp. Γ_i), $i \in \{h, hp, p\}$. Thus $\sigma = (t, \bar{\sigma})$ for any t of $[0, T]$.

We suppose that $\Gamma_p\backslash\Gamma_{hp}$ has a non-zero $\mathcal{H}^{n-1}$-measure. We may now consider the Hilbert space

$V = \{v \in H^1(\Omega_p), v = 0 \textit{ a.e. on } \Gamma_p\backslash\Gamma_{hp}\}$.

used with the norm $\|v\|_V = \|\nabla v\|_{L^2(\Omega_p)^n}$, equivalent to the classical $H^1(\Omega_p)$-norm. We denote by $\langle\langle ., .\rangle\rangle$ the pairing between V and V' and by $\langle ., .\rangle$ the pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Furthermore,

$W(0, T) \equiv \{v \in L^2(0, T; H_0^1(\Omega)); \partial_t v \in L^2(0, T; H^{-1}(\Omega))\}$,

endowed with the norm $\|v\|_{W(0,T)} = \left(\|\partial_t v\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|v\|^2_{L^2(0,T;H_0^1(\Omega))}\right)^{1/2}$. We recall that $W(0, T) \subset \mathcal{C}([0, T]; L^2(\Omega))$.

The function sgn_μ denotes the Lipschitzian and bounded approximation of sgn given for any positive μ and any nonnegative real x by

$\mathrm{sgn}_\mu(x) = \min\left(\frac{x}{\mu}, 1\right)$ and $\mathrm{sgn}_\mu(-x) = -\mathrm{sgn}_\mu(x)$.

Lastly, to simplify the notation, we set for i in $\{h, p\}$:

$G_i(u, v) = g_i(t, x, u) - K_i(v)\mathrm{div}\mathbf{B}_i$ and $F_i(u, v) = \mathrm{sgn}(u - v)(K_i(u) - K_i(v))$,

$L_i(u, v, w) = |u - v|\partial_t w - F_i(u, v)\mathbf{B}_i \cdot \nabla w - \mathrm{sgn}(u - v)G_i(u, v)w$.

and

$L(u, v, w) = L_p(u, v, w)\mathbb{I}_{\Omega_p}(x) + L_h(u, v, w)\mathbb{I}_{\Omega_h}(x)$,

$\mathbf{K}(x, u) = K_p(u)\mathbf{B}_p\mathbb{I}_{\Omega_p}(x) + K_h(u)\mathbf{B}_h\mathbb{I}_{\Omega_h}(x)$,

$\mathcal{F}_h(u, v, w) = \frac{1}{2}\{|K_h(u) - K_h(v)| - |K_h(w) - K_h(v)| + |K_h(u) - K_h(w)|\}$.

3 Statement of uniqueness

3.1 Global definition

We provide a definition of (1)–(2) by considering that (1) can be viewed as a quasilinear parabolic equation that *strongly degenerates* on a fixed subdomain. So we refer to [2,3] to propose a weak formulation through a global entropy inequality on the whole Q. That is why it will be said that:

Definition 1 A measurable function u a solution to (1)–(2) if,

$$u \in L^\infty(Q),\ a < -\|u\|_\infty \le \|u\|_\infty < b,\ \phi(u) \in L^2(0, T; V), \tag{4}$$

$\forall\zeta \in \mathcal{D}(Q), \zeta \ge 0, \forall\kappa \in \mathbb{R}$,

$$\begin{aligned}&\int_Q L(u, \kappa, \zeta)\mathrm{d}x\mathrm{d}t - \int_{Q_p} \nabla|\phi(u) - \phi(\kappa)| \cdot \nabla\zeta \mathrm{d}x\mathrm{d}t\\ &\quad - \int_{\Sigma_{hp}} \{K_h(\kappa)\mathbf{B}_h - K_p(\kappa)\mathbf{B}_p\} \cdot \nu_h \mathrm{sgn}(\phi(u) - \phi(\kappa))\zeta \mathrm{d}\sigma \ge 0,\end{aligned} \tag{5}$$

$\forall\zeta \in L^1(\Sigma_h\backslash\Sigma_{hp}), \zeta \ge 0, \forall\kappa \in \mathbb{R}$

$$\operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_h\backslash\Sigma_{hp}} \mathcal{F}_h(u(\sigma + \tau\nu_h), 0, \kappa)\mathbf{B}_h(\bar{\sigma}) \cdot \nu_h\zeta \mathrm{d}\sigma \le 0, \tag{6}$$

$$\operatorname{ess}\lim_{t\to 0^+}\int_\Omega |u(t,x)-u_0(x)|\mathrm{d}x = 0. \tag{7}$$

Remark 1

(i) In (5), the term $\mathrm{sgn}(\phi(u)-\phi(\kappa))$ on Σ_{hp} has to be understood in the sense of the trace of $\phi(u)$ on Σ_{hp} from the side of Q_p.

(ii) We consider (5) with $\kappa = b$ and with $\kappa = a$. By comparing the two resulting inequalities we remark that the boundary integrals and all the κ-dependent terms collapse and it results for any ζ in $H_0^1(Q)$ that

$$\int_Q \left(u\partial_t\zeta - \left(\mathbb{I}_{\Omega_p}\nabla\phi(u) + \mathbf{K}(x,u)\right)\cdot\nabla\zeta - g(t,x,u)\zeta\right)\mathrm{d}x\mathrm{d}t = 0. \tag{8}$$

Hence u is a weak solution since, in the sense of distributions on Q,

$$\partial_t u - \operatorname{div}\left(\mathbb{I}_{\Omega_p}\nabla\phi(u) + \mathbf{K}(x,u)\right) + g(t,x,u) = 0.$$

3.2 Study on the hyperbolic zone

We derive from (5) and (6) an entropy inequality on the hyperbolic domain that will be the starting point for establishing a time-Lipschitzian dependence in $L^1(\Omega_h)$ of a weak solution to (1)–(2) with respect to the corresponding initial data. To do so as in [3] and by using (5) and (6), we state first that for any κ in $\mathbb{R}$ and any φ of $\mathcal{D}(]0,T[\times\mathbb{R}^n)$, $\varphi \geq 0$:

$$\begin{aligned}-\int_{Q_h} L_h(u,\kappa,\varphi)\mathrm{d}x\mathrm{d}t \leq{}& \operatorname{ess}\lim_{\tau\to 0^-}\int_{\Sigma_{hp}} |K_h(u(\sigma+\tau\nu_h)) - K_h(\kappa)|\mathbf{B}_h(\bar{\sigma})\cdot\nu_h\varphi(\sigma)\mathrm{d}\sigma\\ &-\operatorname{ess}\lim_{\tau\to 0^-}\int_{\Sigma_h\setminus\Sigma_{hp}} |K_h(u(\sigma+\tau\nu_h)) - K_h(0)|\mathbf{B}_h\cdot\nu_h\varphi\mathrm{d}\sigma + |K_h(\kappa)\\ &-K_h(0)|\int_{\Sigma_h\setminus\Sigma_{hp}}\mathbf{B}_h\cdot\nu_h\varphi\mathrm{d}\sigma.\end{aligned}$$

But due to (3) and to the monotonicity of K_h, the first integral in the right-hand side is non-positive. We deduce that, if u is a measurable and bounded function on Q satisfying (5) and (6), then for any κ in $\mathbb{R}$ and any φ of $\mathcal{D}(]0,T[\times\mathbb{R}^n)$, $\varphi\geq 0$,

$$\begin{aligned}-\int_{Q_h} L_h(u,\kappa,\varphi)\mathrm{d}x\mathrm{d}t \leq{}& -\operatorname{ess}\lim_{\tau\to 0^-}\int_{\Sigma_h\setminus\Sigma_{hp}} |K_h(u(\sigma+\tau\nu_h)) - K_h(0)|\mathbf{B}_h\cdot\nu_h\varphi\mathrm{d}\sigma\\ &+|K_h(\kappa)-K_h(0)|\int_{\Sigma_h\setminus\Sigma_{hp}}\mathbf{B}_h\cdot\nu_h\varphi\mathrm{d}\sigma.\end{aligned} \tag{9}$$

In order to use the method of doubling variables, we need a technical result based on properties of mollifiers which has already been pointed out in [4] (or in [5, Chapt. 2]), [6]:

Lemma 1 *Let u be a measurable and bounded function on Q_h such that (9) holds. Then for any continuous function φ on $Q_h\cup\Sigma_h$*

$$\begin{aligned}&\lim_{\delta\to 0^+}\int_{Q_h}\int_{\Sigma_h\setminus\Sigma_{hp}} |K_h(u(p)) - K_h(0)|\mathbf{B}_h(\bar{\tilde{\sigma}})\cdot\nu_h\varphi\left(\frac{\tilde{p}_{|\Sigma}+p}{2}\right)\mathcal{W}_\delta\left(\tilde{p}_{|\Sigma}-p\right)\mathrm{d}\tilde{\sigma}\mathrm{d}p\\ &\quad=\frac{1}{2}\operatorname{ess}\lim_{\tau\to 0^-}\int_{\Sigma_h\setminus\Sigma_{hp}} |K_h(u(\sigma+\tau\nu_h)) - K_h(0)|\mathbf{B}_h(\bar{\sigma})\nu_h\varphi(\sigma)\mathrm{d}\sigma\end{aligned}$$

and,

$$\begin{aligned}&\lim_{\delta\to 0^+}\int_{Q_h}\operatorname{ess}\lim_{\tau\to 0^-}\int_{\Sigma_h\setminus\Sigma_{hp}} |K_h(u(\sigma+\tau\nu_h)) - K_h(0)|\mathbf{B}_h(\bar{\sigma})\cdot\nu_h\varphi(\tfrac{p_{|\Sigma}+\tilde{p}}{2})\mathcal{W}_\delta(p_{|\Sigma}-\tilde{p})\mathrm{d}\sigma\mathrm{d}\tilde{p}\\ &\quad=\frac{1}{2}\operatorname{ess}\lim_{\tau\to 0^-}\int_{\Sigma_h\setminus\Sigma_{hp}} |K_h(u(\sigma+\tau\nu_h)) - K_h(0)|\mathbf{B}_h(\bar{\sigma})\cdot\nu_h\varphi(\sigma)\mathrm{d}\sigma,\end{aligned}$$

where $(\mathcal{W}_\delta)_{\delta>0}$ *is defined on* $\mathbb{R}^{n+1}$ *through:*

$$\forall\delta > 0,\ \forall p = (t, x_1, .., x_n) \in \mathbb{R}^{n+1},\ \mathcal{W}_\delta(p) = \rho_\delta(t)\prod_{i=1}^{n}\rho_\delta(x_i),$$

where $(\rho_\delta)_{\delta>0}$ *is a standard sequence of mollifiers on* $\mathbb{R}$.

Now from (9) and Lemma 1 we derive:

Theorem 1 *Let* u_1 *and* u_2 *be two weak solutions to* (1)–(2) *for initial data* $u_{0,1}$ *and* $u_{0,2}$*, respectively. Then for a.e. t in* $]0,\ T[$, $\displaystyle\int_{\Omega_h}|u_1(t,.) - u_2(t,.)|\mathrm{d}x \le \mathrm{e}^{M'_{g_h}t}\int_{\Omega_h}|u_{0,1} - u_{0,2}|\mathrm{d}x.$

Proof We choose in (9) for u_1 written in variables $p = (t, x)$,

$\kappa = u_2(\tilde{t}, \tilde{x})$,

and in (9) for u_2 in variables $\tilde{p} = (\tilde{t}, \tilde{x})$, $\kappa = u_1(t, x)$. Furthermore in (9) for u_1,

$$\varphi(p, \tilde{p}) = \zeta\left(\frac{p+\tilde{p}}{2}\right)\mathcal{W}_\delta(p - \tilde{p}),$$

where ζ belongs to $\mathcal{D}(]0, T[\times\mathbb{R}^n)$, $\zeta \ge 0$ and similarly in (9) for u_2. We integrate over Q_h on the $\tilde{p}$ variables for u_1 and on the p variables for u_2. We add up. Through classical techniques we pass to the limit with δ on the left-hand side. The right-hand side goes to 0 with δ, thanks to Lemma 1 for u_1 and u_2. This results in:

$$-\int_{Q_h}\{|u_1 - u_2|\partial_t\zeta - |K_h(u_1) - K_h(u_2)|\mathbf{B}_h\cdot\nabla\zeta\}\mathrm{d}x\mathrm{d}t \le -\int_{Q_h}\mathrm{sgn}(u_1 - u_2)(g_h(t, x, u_1) - g_h(t, x, u_2))\zeta\,\mathrm{d}x\mathrm{d}t.$$

For $\zeta \equiv \alpha\psi$, where α belongs to $\mathcal{D}(0, T)$, $\alpha \ge 0$ and ψ to $\mathcal{D}(\mathbb{R}^n)$, $\psi \ge 0$, $\psi \equiv 1$ on Q_h, the Lipschitz condition for g_h provides:

$$-\int_{Q_h}|u_1 - u_2|\alpha'(t)\mathrm{d}x\mathrm{d}t \le M'_{g_h}\int_{Q_h}|u_1 - u_2|\alpha(t)\mathrm{d}x\mathrm{d}t.$$

The conclusion follows from Gronwall's Lemma. □

3.3 Study in the parabolic zone

On Q_p, we characterize a solution to (1)–(2) through a variational equality including the contribution of data entering from the hyperbolic zone. Indeed:

Proposition 1 *Let u be a weak solution to* (1)–(2). *Then* $\partial_t u$ *belongs to* $L^2(0, T; V')$. *Furthermore, for any v in* $L^2(0, T; V)$,

$$\int_0^T\langle\langle\partial_t u, v\rangle\rangle\mathrm{d}t + \int_{Q_p}\left(\nabla\phi(u) + K_p(u)\mathbf{B}_p\right)\cdot\nabla v\mathrm{d}x\mathrm{d}t + \int_{Q_p}g_p(t, x, u)v\mathrm{d}x\mathrm{d}t$$
$$+\,\mathrm{ess}\lim_{\tau\to 0^-}\int_{\Sigma_{hp}}K_h(u(\sigma + \tau\nu_h))\mathbf{B}_h\cdot\nu_h v\mathrm{d}\sigma = 0. \tag{10}$$

Proof Because of a density argument (8) is still true for any ζ in $\mathcal{D}(0, T; H_0^1(\Omega))$. Now let φ be given in $\mathcal{D}(0, T; V)$. We consider $\hat{\varphi}$ to be an extension of φ to $\mathcal{D}(0, T; H_0^1(\Omega))$ and we take $\zeta = \hat{\varphi}\xi_\varrho$ in (8) where ξ_ϱ belongs to $W^{1,+\infty}(\Omega)$, $0 \le \xi_\varrho \le 1$, and fulfills for any positive ϱ:

$$\xi_\varrho(x) = \begin{cases} 1 \text{ if } x \in \bar{\Omega}_p, \\ 0 \text{ if } x \in \Omega_h,\ \mathrm{dist}(x, \Gamma_{hp}) \ge \varrho, \\ \|\nabla\xi_\varrho\|_\infty \le C/\varrho. \end{cases}$$

To pass to the limit when ϱ goes to 0^+, we claim as in [3] that

$$\lim_{\varrho\to 0^+}\int_{Q_h} K_h(u)\hat{\varphi}\mathbf{B}_h\cdot\nabla\xi_\varrho\mathrm{d}x\mathrm{d}t = \mathrm{ess}\lim_{\tau\to 0^-}\int_{\Sigma_{hp}} K_h(u(\sigma+\tau\nu_h))\varphi\mathbf{B}_h\cdot\nu_h\mathrm{d}\sigma.$$

This way, for any φ in $\mathcal{D}(0,T;V)$, $\varphi\geq 0$,

$$\int_{Q_p} u\partial_t\varphi\mathrm{d}x\mathrm{d}t = \int_{Q_p}\big(\nabla\phi(u)+K_p(u)\mathbf{B}_p\big)\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t + \int_{Q_p} g_p(t,x,u)\varphi\mathrm{d}x\mathrm{d}t$$
$$+\,\mathrm{ess}\lim_{\tau\to 0^-}\int_{\Sigma_{hp}} K_h(u(\sigma+\tau\nu_h))\varphi\mathbf{B}_h\cdot\nu_h\mathrm{d}\sigma. \quad (11)$$

Since u is bounded and $\phi(u)$ belongs to $L^2(0,T;V)$ we may argue—thanks to the Trace Theorem—that there exists a constant C such that

$$\forall\varphi\in\mathcal{D}(0,T;V),\ \left|\int_{Q_p} u\partial_t\varphi\mathrm{d}x\mathrm{d}t\right|\leq C\|\varphi\|_{L^2(0,T;V)},$$

which ensures that $\partial_t u$ belongs to $L^2(0,T;V')$ (see Appendix of [7]). Thus,

$$\forall\varphi\in\mathcal{D}(0,T;V),\ -\int_{Q_p} u\partial_t\varphi\mathrm{d}x\mathrm{d}t = \int_0^T\langle\langle\partial_t u,\varphi\rangle\rangle\mathrm{d}t.$$

By a density argument we may rewrite (11) with φ in $L^2(0,T;V)$ and (10) follows which completes the proof of Proposition 1. □

3.4 The uniqueness theorem

Let u_1 and u_2 be two solutions to (1)–(2) having the same initial data on the hyperbolic zone. Because of Theorem 1, we are sure that $u_1=u_2$ a.e. on Q_h. On the parabolic zone, the uniqueness proof uses a method of doubling only the time variable and to deal with the convective terms, we need to assume that

$$\begin{array}{l} K_p\circ\phi^{-1} \text{ is Hölder continuous on } \phi([a,b]) \\ \quad\text{with a constant } \mathcal{C} \text{ and exponent } \theta\geq 1/2. \end{array} \quad (12)$$

Then we may assert that:

Theorem 2 *Under (12) the problem (1)–(2) admits at most one weak solution. Besides, if u_1 and u_2 are two weak solutions corresponding to initial data $u_{0,1}$ and $u_{0,2}$ such that $u_{0,1}=u_{0,2}$ a.e. on Ω_h, then*

$$\textit{for a.e. } t \textit{ in }]0,T[,\quad \int_\Omega |u_1(t,.)-u_2(t,.)|\mathrm{d}x\leq \mathrm{e}^{M'_{g_p}t}\int_{\Omega_p}|u_{0,1}-u_{0,2}|\mathrm{d}x.$$

Proof In (10) for u_1 and written in variables (t,x) we consider $v(t,\tilde{t},x)=\mathrm{sgn}_\mu(\phi(u_1)(t,x)-\phi(u_2)(\tilde{t},x))\alpha_\delta(\tilde{t},t)$ while in (10) for u_2 written in variables $(\tilde{t},x)$, we take the test function $-v(t,\tilde{t},x)$. For any positive δ,

$$\alpha_\delta(\tilde{t},t)=\gamma((t+\tilde{t})/2)\rho_\delta\big((t-\tilde{t})/2\big),$$

where γ is a nonnegative element of $\mathcal{D}(0,T)$ and δ is small enough for α_δ to belong to $\mathcal{D}(]0,T[\times]0,T[)$. By adding up, we have:

$$\int_0^T \int_0^T \langle\langle \partial_t u_1 - \partial_{\tilde t}\tilde u_2, \mathrm{sgn}_\mu(\phi(u_1)-\phi(\tilde u_2))\rangle\rangle \alpha_\delta \mathrm{d}t\mathrm{d}\tilde t$$
$$+\int_{]0,T[\times Q_p} \nabla(\phi(u_1)-\phi(\tilde u_2))\cdot\nabla \mathrm{sgn}_\mu(\phi(u_1)-\phi(\tilde u_2))\alpha_\delta \mathrm{d}x\mathrm{d}t\mathrm{d}\tilde t$$
$$+\int_{]0,T[\times Q_p} (K_p(u_1)-K_p(\tilde u_2))\mathbf{B}_p\cdot\nabla \mathrm{sgn}_\mu(\phi(u_1)-\phi(\tilde u_2))\alpha_\delta \mathrm{d}x\mathrm{d}t\mathrm{d}\tilde t$$
$$+\int_{]0,T[\times Q_p} (g_p(t,x,u_1)-g_p(\tilde t,x,\tilde u_2))\mathrm{sgn}_\mu(\phi(u_1)-\phi(\tilde u_2))\alpha_\delta \mathrm{d}x\mathrm{d}t\mathrm{d}\tilde t$$
$$=-\int_0^T \operatorname{ess}\lim_{\tau\to 0^-}\int_{\Sigma_{hp}} K_h(u_1(\sigma+\tau\nu_h))\mathbf{B}_h\cdot\nu_h \mathrm{sgn}_\mu(\phi(u_1)-\phi(\tilde u_2))\alpha_\delta \mathrm{d}\sigma\mathrm{d}\tilde t$$
$$+\int_0^T \operatorname{ess}\lim_{\tau\to 0^-}\int_{\Sigma_{hp}} K_h(u_2(\tilde\sigma+\tau\nu_h))\mathbf{B}_h\cdot\nu_h \mathrm{sgn}_\mu(\phi(u_1)-\phi(\tilde u_2))\alpha_\delta \mathrm{d}\tilde t\mathrm{d}\bar\sigma\mathrm{d}t \tag{13}$$

So as to simplify the notation, we add a "tilde" superscript to any function in the $\tilde t$ variable. We want to pass to the limit in (13) when μ goes to 0^+ and then when δ tends to 0^+. In the first line of the left-hand side, we use for each term an integration-by-parts formula based on a convexity inequality (see e.g. [8], the Mignot–Bamberger Lemma) to obtain:

$$-\int_{]0,T[\times Q_p} \left(\left(\int_{\tilde u_2}^{u_1} \mathrm{sgn}_\mu(\phi(r)-\phi(\tilde u_2))dr\right)\partial_t\alpha_\delta\right)\mathrm{d}x\mathrm{d}t\mathrm{d}\tilde t$$
$$-\int_{]0,T[\times Q_p} \left(\left(\int_{\tilde u_2}^{u_1} \mathrm{sgn}_\mu(\phi(u_1)-\phi(r))dr\right)\partial_{\tilde t}\alpha_\delta\right)\mathrm{d}x\mathrm{d}t\mathrm{d}\tilde t$$

In the third line we write $K_p(u)=K_p\circ\phi^{-1}(\phi(u))$ for u_1 and for $\tilde u_2$. Then due to (12) and to the Young inequality with $p=2$:

$$\int_{]0,T[\times Q_p} (K_p(u_1)-K_p(\tilde u_2))\mathbf{B}_p\cdot\nabla \mathrm{sgn}_\mu(\phi(u_1)-\phi(\tilde u_2))\alpha_\delta \mathrm{d}x\mathrm{d}t\mathrm{d}\tilde t$$
$$\leq \frac{\mathcal{C}^2\|\mathbf{B}_p\|^2_{L^\infty(\Omega_p)^n}}{2}\int_{]0,T[\times Q_p} |\phi(u_1)-\phi(\tilde u_2)|^{2\theta}\alpha_\delta \mathrm{sgn}'_\mu(\phi(u_1)-\phi(\tilde u_2))\mathrm{d}x\mathrm{d}t\mathrm{d}\tilde t$$
$$+\frac{1}{2}\int_{]0,T[\times Q_p} |\nabla(\phi(u_1)-\phi(\tilde u_2))|^2\alpha_\delta \mathrm{sgn}'_\mu(\phi(u_1)-\phi(\tilde u_2))\mathrm{d}x\mathrm{d}t\mathrm{d}\tilde t,$$

where in the right-hand side the second integral vanishes into the diffusion term while, by coming back to the definition of sign_μ, the first one is estimated by

$$C^{st}\int_{]0,T[\times Q_p} \mu^{2\theta-1}\alpha_\delta \mathbb{I}_{[-\mu<\phi(u_1)-\phi(\tilde u_2)<\mu]}\mathrm{d}x\mathrm{d}t\mathrm{d}\tilde t,$$

which goes to 0 with μ as soon as $\theta \geq 1/2$.

Now for the right-hand side of (13), since $u_1=u_2$ a.e. on Q_h then $u_1(\sigma+\tau\nu_h)=u_2(\sigma+\tau\nu_h)$ for any negative τ. However, from (9) used with φ in $\mathcal{D}(Q_h)$, we argue as in [4] (see also [5, chap. 2]) that for any open subset Σ_{loc} of Σ_h there exists Θ in $L^\infty(\Sigma_{\mathrm{loc}})$ such that:

$$\operatorname{ess}\lim_{\tau\to 0^-}\int_{\Sigma_{\mathrm{loc}}} K_h(u(\sigma+\tau\nu_h))\mathbf{B}_h\cdot\nu_h\beta\mathrm{d}\sigma=\int_{\Sigma_{loc}}\Theta(\sigma)\beta\mathrm{d}\sigma,$$

for any β in $L^1(\Sigma_{\mathrm{loc}})$. We refer to this relation when $\Sigma_{\mathrm{loc}}=\Sigma_{hp}$. Thus,

$$\operatorname{ess}\lim_{\tau\to 0^-}\int_{\Sigma_{hp}} K_h(u_2(\sigma+\tau\nu_h))\mathbf{B}_h\cdot\nu_h \mathrm{sgn}_\mu(\phi(u_1)(\sigma)-\phi(u_2)(\tilde t,\bar\sigma))\alpha_\delta(\tilde t,t)\mathrm{d}\sigma$$
$$=\int_{\Sigma_{hp}}\Theta(\sigma)\mathrm{sgn}_\mu(\phi(u_1)(\sigma)-\phi(u_2)(\tilde t,\bar\sigma))\alpha_\delta(\tilde t,t)\mathrm{d}\sigma.$$

So that finally we have to consider the term

$$\int_0^T \int_0^T \int_{\Gamma_{hp}} (\Theta(t,\bar{\sigma}) - \Theta(\tilde{t},\bar{\sigma}))\mathrm{sgn}_\mu(\phi(u_1)(\sigma) - \phi(u_2)(\tilde{t},\bar{\sigma}))\alpha_\delta(\tilde{t},t)\mathrm{d}\bar{\sigma}\mathrm{d}t\mathrm{d}\tilde{t}.$$

Eventually, when μ goes to 0^+ in (13) through the Lebesgue-dominated convergence Theorem, we obtain:

$$\begin{aligned}-\int_{]0,T[\times Q_p} |u_1 - u_2|(\partial_t\alpha_\delta + \partial_{\tilde{t}}\alpha_\delta)\mathrm{d}x\mathrm{d}t\mathrm{d}\tilde{t} \le{}& \int_0^T \int_0^T \int_{\Gamma_{hp}} |\Theta(t,\bar{\sigma}) - \Theta(\tilde{t},\bar{\sigma})|\alpha_\delta \mathrm{d}\bar{\sigma}\mathrm{d}t\mathrm{d}\tilde{t} \\ &+ M'_{g_p}\int_{]0,T[\times Q_p} |u_1 - u_2|\alpha_\delta \mathrm{d}x\mathrm{d}t\mathrm{d}\tilde{t} \\ &+ \int_{]0,T[\times Q_p} |g_p(t,x,\tilde{u}_2) - g_p(\tilde{t},x,\tilde{u}_2)|\alpha_\delta \mathrm{d}x\mathrm{d}t\mathrm{d}\tilde{t}.\end{aligned}$$

Now, we return to the definition of α_δ to express the sum $\partial_t\alpha_\delta + \partial_{\tilde{t}}\alpha_\delta$. Then we are able to take the limit with respect to δ through the notion of the Lebesgue points for an integrable function on $]0,T[$. For any γ of $\mathcal{D}(0,T)$, $\gamma \ge 0$, we obtain

$$-\int_{Q_p} |u_1 - u_2|\gamma'(t)\mathrm{d}x\mathrm{d}t \le M'_{g_p}\int_{Q_p} |u_1 - u_2|\gamma(t)\mathrm{d}x\mathrm{d}t.$$

We use Gronwall's Lemma to complete the statement of Theorem 2. □

Remark 2 As a consequence of the proof of Theorem 2 and the preceding statement, we may propose an equivalent definition of (1)–(2) that might read:

- $u \in L^\infty(Q),\ a < -\|u\|_\infty \le \|u\|_\infty < b,\ \phi(u) \in L^2(0,T;V)$,
- $\forall\zeta \in \mathcal{D}(Q_h),\ \zeta \ge 0,\ \forall\kappa \in \mathbb{R},\ \int_{Q_h} L(u,\kappa,\zeta)\mathrm{d}x\mathrm{d}t \ge 0.$
- $\forall v \in L^2(0,T;V)$,

$$\begin{aligned}&\int_0^T \langle\langle\partial_t u, v\rangle\rangle\mathrm{d}t + \int_{Q_p}\left(\nabla\phi(u) + K_p(u)\mathbf{B}_p\right)\cdot\nabla v\mathrm{d}x\mathrm{d}t + \int_{Q_p} g_p(t,x,u)v\mathrm{d}x\mathrm{d}t \\ &\quad+ \mathrm{ess}\lim_{\tau\to 0^-}\int_{\Sigma_{hp}} K_h(u(\sigma+\tau\nu_h))\mathbf{B}_h\cdot\nu_h v\mathrm{d}\sigma = 0.\end{aligned}$$

- $\forall\zeta \in H_0^1(Q),\ \int_Q \left(u\partial_t\zeta - \left(\mathbb{I}_{\Omega_p}\nabla\phi(u) + \mathbf{K}(x,u)\right)\cdot\nabla\zeta - g(t,x,u)\zeta\right)\mathrm{d}x\mathrm{d}t = 0.$
- $\forall\zeta \in L^1_+(\Sigma_h\backslash\Sigma_{hp}),\ \forall\kappa \in \mathbb{R}$,

$$\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_h\backslash\Sigma_{hp}} \mathcal{F}_h(u(\sigma+\tau\nu_h),0,\kappa)\mathbf{B}_h(\bar{\sigma})\cdot\nu_h\zeta\mathrm{d}\sigma \le 0.$$

- $\operatorname*{ess\,lim}_{t\to 0^+}\int_\Omega |u(t,x) - u_0(x)|\mathrm{d}x = 0.$

4 The existence property

4.1 The second-order problem

We approximate a weak solution to (1)–(2) through a sequence of solutions to viscous problems deduced from (1)–(2) by adding a diffusion term in accordance with the proposed physical modeling of two layers in the subsoil with different geological characteristics. So for any positive ϵ, we are first interested in the uniqueness and existence of a measurable and bounded function u_ϵ on Q satisfying

$$\partial_t u_\epsilon - \sum_{j=1}^{n} \partial_{x_j}(\lambda_\epsilon(x)\partial_{x_i}\phi_\epsilon(u_\epsilon) + K_j(x, u_\epsilon)) + g(t, x, u_\epsilon) = 0 \text{ on } Q, \tag{14}$$

$$u_\epsilon = 0 \text{ on }]0, T[\times\partial\Omega,\ u_\epsilon(0, .) = u_0 \text{ on } \Omega, \tag{15}$$

with $\lambda_\epsilon(x) = \mathbb{I}_{\Omega_p}(x) + \epsilon\mathbb{I}_{\Omega_h}(x)$ and $\phi_\epsilon(u_\epsilon) = \phi(u_\epsilon) + \epsilon u_\epsilon$.

We define $\mathcal{N} = N_1 + N_2$, $N_1 = \sum_{i\in\{h,p\}} M'_{g_i}$, $N_2 = \sum_{i\in\{h,p\}} \mathcal{K}'_i \|\text{div}\mathbf{B}_i\|_{L^\infty(\Omega_i)}$, $N_3 = \sum_{i\in\{h,p\}} \text{ess sup}_{[0,T]\times\bar{\Omega}_i} (g_i(t, x, b) + K_i(b)\text{div}\mathbf{B}_i)^-$ and $N_4 = -\sum_{i\in\{h,p\}} \max_{[0,T]\times\bar{\Omega}_h}(g_i(t, x, a) + K_i(a)\text{div}\mathbf{B}_i)^+$. This way,

$$\exists! m_0 \in \mathbb{R}^+,\ m_0 e^{\mathcal{N}T} + \frac{N_3}{\mathcal{N}}(e^{\mathcal{N}T} - 1) = b,$$

$$\exists! n_0 \in \mathbb{R}^-,\ n_0 e^{\mathcal{N}T} + \frac{N_4}{\mathcal{N}}(e^{\mathcal{N}T} - 1) = a.$$

We choose an initial datum u_0 in $L^\infty(\Omega)$ such that

$$\text{ess}\sup_{\Omega} u_0 \le m_0 \ \text{ and } \ \text{ess}\inf_{\Omega} u_0 \ge n_0$$

and we introduce the nonnegative and nondecreasing time-dependent function

$$M_1 \colon t \in [0, T] \to M_1(t) = m_0\, e^{\mathcal{N}t} + \frac{N_3}{\mathcal{N}}(e^{\mathcal{N}t} - 1), \tag{16}$$

and the non-positive and non-increasing function

$$M_2 \colon t \in [0, T] \to M_2(t) = n_0\, e^{\mathcal{N}t} + \frac{N_4}{\mathcal{N}}(e^{\mathcal{N}t} - 1), \tag{17}$$

so that $M_1(T) = b \ge m_0$ and $M_2(T) = a \le n_0$.

We investigate the behavior of the sequence $(u_\epsilon)_{\epsilon>0}$ when ϵ goes to 0^+. With this view, in order to deal with bounded solutions, we need the following assumptions on K_h and K_p *a.e.* on Γ_{hp}:

$$\forall r \in [m_0, b],\ K_h(M_1(r))\mathbf{B}_h \cdot \nu_h \ge K_p(M_1(r))\mathbf{B}_p \cdot \nu_h, \tag{18}$$

$$\forall r \in [a, n_0],\ K_h(M_2(r))\mathbf{B}_h \cdot \nu_h \le K_p(M_2(r))\mathbf{B}_p \cdot \nu_h, \tag{19}$$

where M_1 and M_2 are defined by (16) and (17). From this we may state the first main theorem of this section:

Proposition 2 *Under* (18) *and* (19) *there exists a unique solution* u_ϵ *to* (14)–(15) *in* $W(0, T) \cap L^\infty(Q)$ *such that*

$$\forall t \in [0, T],\ M_2(t) \le u_\epsilon(t, .) \le M_1(t) \text{ a.e. in } \Omega, \tag{20}$$

$$u_\epsilon(0, .) = u_0 \text{ a.e. in } \Omega, \tag{21}$$

satisfying the variational equality for any v *in* $H_0^1(\Omega)$ *and for a.e.* t *in* $]0, T[$:

$$\langle \partial_t u_\epsilon, v\rangle + \int_\Omega ((\lambda_\epsilon(x)\nabla\phi_\epsilon(u_\epsilon) + \mathbf{K}(x, u_\epsilon)) \cdot \nabla v + g(t, x, u_\epsilon)v)\text{d}x = 0. \tag{22}$$

We recall that, since u_ϵ belongs to $W(0, T)$, for any t in $[0, T]$, $u_\epsilon(t, .)$ is an element of $L^2(\Omega)$; that gives a meaning to $u_\epsilon(0, .)$ a.e. in Ω.

Proof (a) In a first step, we define for any real a, b, c, $\mathcal{B}(a, b, c) = \max\{a, \min\{b, c\}\}$ and we introduce the modified problem for a fixed positive ϵ:

$$\begin{cases} \text{Find } u_\epsilon \text{ in } W(0, T) \text{ such that a.e. on }]0, T[\text{ and for all } v \text{ in } H_0^1(\Omega), \\ \langle \partial_t u_\epsilon, v\rangle + \int_\Omega \big((\lambda_\epsilon \phi'_\epsilon(u^\star_\epsilon)\nabla u_\epsilon + \mathbf{K}(x, u^\star_\epsilon)) \cdot \nabla v + g(t, x, u^\star_\epsilon)v\big)\, \text{d}x = 0, \\ u_\epsilon(0, .) = u_0 \text{ a.e. in } \Omega, \end{cases} \tag{23}$$

where $u_\epsilon^\star(t,x) = \mathcal{B}(M_2(t), u_\epsilon(t,x), M_1(t))$. Indeed (20)–(22) is equivalent to (23) since, if (23) has a solution u_ϵ, then u_ϵ satisfies (20): let u_ϵ be a solution to (23). The majoration for u_ϵ in (20) is obtained by considering in (23) the test function $v_\mu = \mathrm{sgn}_\mu(u_\epsilon - M_1(t))^+$ and by integrating over $]0,s[$, for any s of $]0,T]$. For the evolution term, one adds and subtracts $\langle \partial_t M_1(t), v_\mu\rangle$. Then we integrate using a Mignot–Bamberger formula (see [8]). Also, because of the Green formula and since ∇v_μ is supported on $\{u_\epsilon > M_1\}$, we write the for the convective term (with $Q_s =]0,s[\times\Omega$):

$$\int_{Q_s} \mathbf{K}(x,u_\epsilon^\star)\cdot\nabla v_\mu \mathrm{d}x\mathrm{d}t = \int_{Q_s} \mathbf{K}(x,M_1)\cdot\nabla v_\mu \mathrm{d}x\mathrm{d}t = -\sum_{i\in\{h,p\}}\int_{Q_{i,s}} K_i(M_1(t))\mathrm{div}\mathbf{B}_i v_\mu \mathrm{d}x\mathrm{d}t$$
$$+\int_{\Sigma_{hp}} (K_h(M_1(t))\mathbf{B}_h\cdot\nu_h - K_p(M_1(t))\mathbf{B}_p\cdot\nu_h)v_\mu \mathrm{d}\sigma,$$

where, due to (18), the second line in the left-hand side is nonnegative. The diffusive term being also nonnegative, we have upon μ going to 0^+:

$$\int_\Omega (u_\epsilon(s,x) - M_1(s))^+\mathrm{d}x + \int_{Q_s}\left(M_1'(t) + \sum_{i\in\{h,p\}} G_i(M_1,M_1)\mathbb{I}_{\Omega_i}\right)\mathrm{sgn}(u_\epsilon - M_1(t))^+\mathrm{d}x\mathrm{d}t \leq 0.$$

Returning to the definition of M_1, we make sure that a.e. on Q,

$$\begin{aligned} M_1'(t) + \textstyle\sum_{i\in\{h,p\}} G_i(M_1,M_1)\mathbb{I}_{\Omega_i} &= \mathcal{N}M_1(t) + N_3 + \textstyle\sum_{i\in\{h,p\}} G_i(M_1,M_1)\mathbb{I}_{\Omega_i} \\ &\geq \mathcal{N}M_1(t) + N_3 - \mathcal{N}M_1(t) + \mathcal{N}b + \textstyle\sum_{i\in\{h,p\}}(g_i(t,x,b) \\ &\quad + K_i(b)\mathrm{div}\mathbf{B}_i))\mathbb{I}_{\Omega_i} \geq 0. \end{aligned}$$

The conclusion follows immediately. The reasoning for the minoration in (20) is similar—with the test function $v_\mu = -\mathrm{sgn}_\mu(u_\epsilon - M_2(t))^-$ in (23)—and uses (19) to ensure that the integral along the interface is nonnegative.

Remark 3 The previous calculations highlight the fact that if

$$\sum_{i\in\{h,p\}} (g_i(.,.,n_0) - K_i(n_0)\mathrm{div}\mathbf{B}_i) \leq 0 \quad \text{and} \quad \sum_{i\in\{h,p\}} (g_i(.,.,m_0) - K_i(m_0)\mathrm{div}\mathbf{B}_i) \geq 0,$$

then $[n_0, m_0]$ is an invariant region for u_ϵ, in the sense that if $n_0 \leq u_0 \leq m_0$ a.e. in Ω then $n_0 \leq u_\epsilon \leq m_0$ a.e. in Q. This special framework may be derived from the general one by taking $\mathcal{N} = 0$, so that $b = m_0$, $a = n_0$, $N_3 = N_4 = 0$. This way, for any t in $[0,T]$, $M_1(t) = b$ and $M_2(t) = a$ and (18) and (19),

$$K_h(b)\mathbf{B}_h\cdot\nu_h \geq K_p(b)\mathbf{B}_p\cdot\nu_h,$$
$$K_h(a)\mathbf{B}_h\cdot\nu_h \leq K_p(a)\mathbf{B}_p\cdot\nu_h,$$

which are easier to be satisfied.

The existence property for (14)–(15) is reduced to an existence result for (23). We use the Schauder–Tychonoff fixed-point Theorem which leads, for a given w in $W(0,T)$, to the linearized problem:

$$\begin{cases} \text{find } U \text{ in } W(0,T) \text{ such that for any } v \text{ in } H_0^1(\Omega) \text{ and a.e. in }]0,T[, \\ \langle \partial_t U, v\rangle + \displaystyle\int_\Omega \left((\lambda_\epsilon \phi_\epsilon'(w^\star)\nabla U + \mathbf{K}(x,w^\star))\cdot\nabla v + g(t,x,w^\star)v\right)\mathrm{d}x = 0, \\ U(0,.) = u_0. \end{cases} \tag{24}$$

It is well known that (24) has a unique solution. Thus we may define the operator

$$\mathcal{T} : W(0,T) \to W(0,T)$$
$$w \to U \equiv \mathcal{T}(w)$$

where U is the unique solution to (24). In addition, using $v = U$ in (24), we argue that $\|U\|_{L^2(0,T;H_0^1(\Omega))} \leq C_1$. The former estimate and the definition of the norm in $L^2(0,T;H^{-1}(\Omega))$ entail, by reference to (24), that

$\|\partial_t U\|_{L^2(0,T;H^{-1}(\Omega))} \le C_2$, for constants C_1 and C_2 independent of w (but depending on ϵ). Hence with $C_3 = (C_1^2 + C_2^2)^{1/2}$, we may say that

$$\mathcal{C} = \{U \in W(0,T), \|U\|_{W(0,T)} \le C_3,\ U(0,.) = u_0 \text{ a.e. in } \Omega\}$$

is a convex set, weakly compact in $W(0,T)$, and such that $\mathcal{T}(\mathcal{C}) \subset \mathcal{C}$. As $\mathcal{C}$ is a metric for the weak topology $\sigma(W(0,T), W'(0,T))$, we consider $(w_n)_n$ to be a sequence converging toward w weakly in $W(0,T)$ in order to establish that $(\mathcal{T}(w_n))_n$ weakly converges toward $\mathcal{T}(w)$ in $W(0,T)$. For any n, we set $U_n = \mathcal{T}(w_n)$. Since $(U_n)_n$ is uniformly bounded in $W(0,T)$ with respect to w_n–that is, uniformly with respect to n–there exists a U in $W(0,T)$, such that, up to a subsequence, $(U_n)_n$ goes to U weakly in $W(0,T)$, strongly in $L^2(Q)$ and $U_n(0,.)$ goes to $U(0,.)$ weakly in $L^2(\Omega)$. Consequently $U(0,.) = u_0$ a.e. in Ω and thus, by taking the limit with respect to n in (24), we prove that $U = \mathcal{T}(w)$ and the whole sequence $(U_n)_n$ goes to $\mathcal{T}(w)$. Eventually, $\mathcal{T}$ has at least one fixed point u_ϵ that is a solution to (23).

The statement regarding the uniqueness for (20)–(22) uses a Holmgren-type duality method. Let u and $\widehat{u}$ be two weak solutions. For any t of $[0,T]$, we consider $z(t,.)$ (resp. $\widehat{z}(t,.)$) in $H_0^1(\Omega)$ such that for all v in $H_0^1(\Omega)$, for all t in $[0,T]$:

$$\int_\Omega \lambda_\epsilon \nabla z \cdot \nabla v \mathrm{d}x = \int_\Omega u v \mathrm{d}x \ \left(\text{resp. } \int_\Omega \lambda_\epsilon \nabla \widehat{z} \cdot \nabla v \mathrm{d}x = \int_\Omega \widehat{u} v \mathrm{d}x\right). \tag{25}$$

It should be noted that, since $\partial_t u$ (resp. $\partial_t \widehat{u}$) belongs to $L^2(0,T,V')$, we are able to define, for a.e. t in $]0,T[$, $\partial_t z$ (resp. $\partial_t \widehat{z}$) in $H_0^1(\Omega)$ characterized by the variational equality for any v in $H_0^1(\Omega)$ and t in $[0,T]$

$$\int_\Omega \lambda_\epsilon \nabla \partial_t z \cdot \nabla v \mathrm{d}x = \langle \partial_t u, v\rangle \ \left(\text{resp. } \int_\Omega \lambda_\epsilon \nabla \partial_t \widehat{z} \cdot \nabla v \mathrm{d}x = \langle \partial_t \widehat{u}, v\rangle\right). \tag{26}$$

First we take $v = z - \widehat{z}$ in (22) for u and for $\widehat{u}$ and in (26). We integrate from 0 and s, s in $[0,T]$, to obtain

$$\begin{aligned}\int_{Q_s} \lambda_\epsilon \nabla \partial_t (z-\widehat{z}) \cdot \nabla(z-\widehat{z}) \mathrm{d}x\mathrm{d}t + \int_{Q_s} \lambda_\epsilon \nabla(\phi_\epsilon(u) - \phi_\epsilon(\widehat{u})) \cdot \nabla(z-\widehat{z})\mathrm{d}x\mathrm{d}t \\ = -\int_{Q_s} (\mathbf{K}(x,u) - \mathbf{K}(x,\widehat{u})) \cdot \nabla(z-\widehat{z})\mathrm{d}x\mathrm{d}t - \int_{Q_s} (g(t,x,u) - g(t,x,\widehat{u}))(z-\widehat{z})\mathrm{d}x\mathrm{d}t.\end{aligned}$$

We observe that

$$\begin{aligned}\int_{Q_s} \lambda_\epsilon \nabla \partial_t (z-\widehat{z}) \cdot \nabla(z-\widehat{z})\mathrm{d}x\mathrm{d}t &= \int_{Q_s} \lambda_\epsilon \partial_t(\nabla(z-\widehat{z})) \cdot \nabla(z-\widehat{z})\mathrm{d}x\mathrm{d}t \\ &= \frac{1}{2}\int_\Omega \lambda_\epsilon |\nabla(z-\widehat{z})|^2(s,.)\mathrm{d}x,\end{aligned}$$

since $\nabla z(0,.) = \nabla \widehat{z}(0,.)$ as a consequence of (25) with $t = 0$.

In addition, we choose $v = \phi_\epsilon(u) - \phi_\epsilon(\widehat{u})$ in (25) to write:

$$\begin{aligned}\int_{Q_s} \lambda_\epsilon \nabla(\phi_\epsilon(u) - \phi_\epsilon(\widehat{u})) \cdot \nabla(z-\widehat{z})\mathrm{d}x\mathrm{d}t &= \int_{Q_s} (u-\widehat{u})(\phi_\epsilon(u) - \phi_\epsilon(\widehat{u}))\mathrm{d}x\mathrm{d}t \\ &\ge \epsilon \|u-\widehat{u}\|^2_{L^2(Q_s)}.\end{aligned}$$

We use the Lipschitz condition for K_i and g_i, $i \in \{h,p\}$. Hence Young's inequality (with $p = 2$) provides, for any s in $[0,T]$:

$$\frac{1}{2}\int_{\Omega}\lambda_\epsilon|\nabla(z-\widehat{z})|^2(s,.)\mathrm{d}x+\epsilon\|u-\widehat{u}\|^2_{L^2(Q_s)}\leq 2\|u-\widehat{u}\|_{L^2(Q_s)}(\max(\|\mathbf{B}_h\|_\infty\mathcal{K}'_h,\|\mathbf{B}_p\|_\infty\mathcal{K}'_p)\|\nabla(z-\widehat{z})\|_{L^2(Q_s)^n}$$
$$+\max(M'_{g_h},M'_{g_p})\|z-\widehat{z}\|_{L^2(Q_s)})$$

The Poincaré inequality (to estimate $\|z-\widehat{z}\|_{L^2(Q_s)}$ with $\|\nabla(z-\widehat{z})\|_{L^2(Q_s)^n}$) and the Young inequality lead to the existence of a constant C such that:

$$\frac{1}{2}\int_{\Omega}\lambda_\epsilon|\nabla(z-\widehat{z})|^2(s,.)\mathrm{d}x\leq C\|\nabla(z-\widehat{z})\|^2_{L^2(Q_s)^n}.$$

The conclusion follows by using Gronwall's Lemma (recall that $z(t,.)$ and $\widehat{z}(t,.)$ belong to $H^1_0(\Omega)$).

Now we point out some *a priori* estimates for the sequence $(u_\epsilon)_{\epsilon>0}$ of viscous Problems (14)–(15)$_{\epsilon>0}$:

Proposition 3 *There exists a constant C, independent of ϵ such that*

$$\|(\lambda_\epsilon)^{1/2}\nabla\widehat{\phi}(u_\epsilon)\|^2_{L^2(Q)^n}+\|(\epsilon\lambda_\epsilon)^{1/2}\nabla u_\epsilon\|^2_{L^2(Q)^n}\leq C \tag{27}$$
$$\|\partial_t u_\epsilon\|_{L^2(0,T;H^{-1}(\Omega))}\leq C, \tag{28}$$

where $\widehat{\phi}(x)=\int_0^x\sqrt{\phi'(\tau)}\mathrm{d}\tau.$

Proof We take $v=u_\epsilon$ in (22) and integrate over $]0,T[$. The resulting equality provides an estimate of $\lambda_\epsilon^{1/2}\nabla\widehat{\phi_\epsilon}(u_\epsilon)$ in $L^2(Q)$-norm. We mention that the convective term is split into two integrals over Q_h and Q_p. Then we write for i in $\{h,p\}$:

$$\int_{Q_i}K_i(u_\epsilon)\mathbf{B}_i\cdot\nabla u_\epsilon\mathrm{d}x\mathrm{d}t=\int_{Q_i}\nabla\left(\int_0^{u_\epsilon}K_i(\tau)\mathrm{d}\tau\right).\mathbf{B}_i\mathrm{d}x\mathrm{d}t$$
$$=-\int_{Q_i}\left(\int_0^{u_\epsilon}K_i(\tau)\mathrm{d}\tau\right)\mathrm{div}\mathbf{B}_i\mathrm{d}x\mathrm{d}t+\int_{\Sigma_{hp}}\left(\int_0^{u_\epsilon}K_i(\tau)\mathrm{d}\tau\right)\mathbf{B}_i\cdot\nu_i\mathrm{d}\sigma.$$

Due to (20), each term in the left-hand side is uniformly bounded with respect to ϵ and (27) follows. Eventually (28) is obtained by coming back to the definition of the norm in $L^2(0,T;H^{-1}(\Omega))$, by using (22) and the estimates (20) and (27), with the same arguments as towards the end of the proof of Proposition 1. □

4.2 The viscous limit

To describe the behavior of the sequence $(u_\epsilon)_{\epsilon>0}$ when ϵ goes to 0^+ on the hyperbolic domain, we take advantage of (20) and of:

Claim 1 ([9]) *Let $\mathcal{O}$ be an open bounded subset of $\mathbb{R}^q$ ($q\geq 1$) and $(u_n)_{n>0}$ a sequence of measurable functions on $\mathcal{O}$ such that,*

$$\exists M>0,\forall n>0,\quad \|u_n\|_{L^\infty(\mathcal{O})}\leq M.$$

Then there exist a subsequence $(u_{\varphi(n)})_{n>0}$ and a measurable function π in $L^\infty(]0,1[\times\mathcal{O})$ such that, for all continuous and bounded functions f on $\mathcal{O}\times]-M,M[$,

$$\forall\xi\in L^1(\mathcal{O}),\quad \lim_{n\to+\infty}\int_{\mathcal{O}}f(x,u_{\varphi(n)})\xi\mathrm{d}x=\int_{]0,1[\times\mathcal{O}}f(x,\pi(\alpha,w))\mathrm{d}\alpha\xi\mathrm{d}x.$$

Such a result has first been applied to the approximation through the artificial-viscosity method of the Cauchy problem in $\mathbb{R}^p$ for conservation laws, as one can establish a uniform L^∞-control of approximate solutions. It has also been applied to the numerical analysis of transport equations since "Finite-Volume" schemes only give an

L^∞-estimate uniformly with respect to the mesh length of the numerical solution [9]. On the parabolic area, estimates (20), (27) and (28) are not sufficient to study the behavior of $(u_\epsilon)_{\epsilon>0}$. That is why we need an additional assumption on ϕ:

$$\phi^{-1} \text{ is Hölder-continuous on } \phi([M_2(T), M_1(T)]) \text{with an exponent } \tau \text{ in }]0,1[. \tag{29}$$

In this framework, we can refer to the arguments put forward in [8, Chapt. 2]. From (28) the sequence $(\partial_t u_\epsilon)_{\epsilon>0}$ remains fixed in a bounded subset of $L^2(0,T;H^{-1}(\Omega_p))$ and due to (20) and (27), the sequence $(\phi(u_\epsilon))_{\epsilon>0}$ is bounded in $L^2(0,T;V)$ uniformly with respect to ϵ. Using that

$$\forall s \in]0,1[,\ L^2(0,T;V) \hookrightarrow L^2(0,T;H^1(\Omega_p)) \hookrightarrow L^2(0,T;W^{s,2}(\Omega_p)),$$

we argue that $u_\epsilon \equiv \phi^{-1}(\phi(u_\epsilon))$ is bounded in $L^{2/\tau}(0,T;W^{\tau s,2/\tau}(\Omega_p))$. The compact embedding of $W^{\tau s,2/\tau}(\Omega_p)$ in $L^{2/\tau}(\Omega_p)$ and the J.L.Lions compactness Theorem [10, p. 57] ensure that $\mathcal{W} \equiv \{v \in L^{2/\tau}(0,T;W^{\tau s,2/\tau}(\Omega_p));\ \partial_t v \in L^2(0,T;H^{-1}(\Omega_p))\}$ is compactly embedded in $L^{2/\tau}(0,T;L^{2/\tau}(\Omega_p))$. Eventually we have:

Proposition 4 *When (29) holds, there exists a measurable function u in $L^\infty(Q)$ with $\phi(u)$ in $L^2(0,T;V)$ and such that up to a subsequence when ϵ goes to 0^+,*

$$u_\epsilon \rightharpoonup u \text{ in } L^\infty(Q) \text{ weak} - \star, \text{ and in } L^q(Q_p),\ 1 \le q < +\infty,$$
$$\nabla\phi(u_\epsilon) \rightharpoonup \nabla\phi(u) \text{ weakly in } L^2(Q_p)^n,\ \epsilon\nabla\phi(u_\epsilon) \to 0^+ \text{ strongly in } L^2(Q_h)^n,$$
$$\lambda_\epsilon \epsilon \nabla u_\epsilon \to 0 \text{ strongly in } L^2(Q_h)^n.$$

We are now able to state the second main theorem of this section:

Theorem 3 *Problem (1)–(2) has a weak solution that is the limit in $L^q(Q)$, $1 \le q < +\infty$ of the whole sequence of solutions to (14)–(15)$_{\epsilon>0}$ when ϵ goes to 0^+.*

Proof We consider the function u highlighted in Proposition 4. Since $(u_{\epsilon|\Omega_h})_{\epsilon>0}$ is uniformly bounded, there exists a subsequence, still labeled $(u_{\epsilon|\Omega_h})_{\epsilon>0}$, and a measurable and bounded function π, called a *process*, on $]0,1[\times Q_h$ such that for any continuous bounded function ψ on $Q_h \times]M_2(T), M_1(T)[$ and ξ in $L^1(Q_h)$

$$\lim_{\epsilon\to 0^+} \int_{Q_h} \psi(t,x,u_\epsilon)\xi \mathrm{d}x\mathrm{d}t = \int_{]0,1[\times Q_h} \psi(t,x,\pi(\alpha,t,x))\xi \mathrm{d}\alpha\mathrm{d}x\mathrm{d}t. \tag{30}$$

We first establish that on Q_h, the *process* π is reduced to $u_{|\Omega_h}$ and second we prove that u is a weak solution to (1)–(2) for initial data u_0. To do so, we return to (22) and for any real κ we take the test function $v_\mu^\epsilon \equiv \mathrm{sgn}_\mu(\phi(u_\epsilon) - \phi(\kappa))\zeta_1\zeta_2$, where ζ_1 belongs to $\mathcal{D}(-T,T)$ and ζ_2 to $\mathcal{D}(\Omega)$, $\zeta_i \ge 0$. We integrate with respect to the time variable and perform the following transformations:
For the evolution term, with $I_\mu(u_\epsilon,\kappa) = \int_\kappa^{u_\epsilon} \mathrm{sgn}_\mu(\phi(\tau) - \phi(\kappa))\mathrm{d}\tau$, through the Mignot–Bamberger integration by parts formula (see [8]),

$$\int_0^T \langle \partial_t u_\epsilon, \mathrm{sgn}_\mu(\phi(u_\epsilon) - \phi(\kappa))\zeta_2\rangle\zeta_1 \mathrm{d}t = -\int_Q I_\mu(u_\epsilon,\kappa)\zeta_2\partial_t\zeta_1 \mathrm{d}x\mathrm{d}t - \int_\Omega I_\mu(u_0,\kappa)\zeta_2\zeta_1(0)\mathrm{d}x.$$

For the diffusion term, we develop the partial derivatives and we use the fact that $\mathrm{sgn}_\mu(.)$ is nondecreasing. Then we have:

$$\int_Q \lambda_\epsilon \nabla\phi_\epsilon(u_\epsilon)\cdot\nabla v_\mu^\epsilon \mathrm{d}x\mathrm{d}t \ge \int_Q \lambda_\epsilon \mathrm{sgn}_\mu(\phi(u_\epsilon) - \phi(\kappa))\nabla\phi(u_\epsilon)\zeta_1 \cdot \nabla\zeta_2 \mathrm{d}x\mathrm{d}t$$
$$+ \int_Q \lambda_\epsilon \epsilon\, \mathrm{sgn}_\mu(\phi(u_\epsilon) - \phi(\kappa))\nabla u_\epsilon \zeta_1 \cdot \nabla\zeta_2 \mathrm{d}x\mathrm{d}t.$$

Now, in order to take the ϵ-limit and then the μ-limit separately in the parabolic and the hyperbolic zones, the convection term is split into two integrals over Q_h and Q_p:

$$\sum_{i\in\{h,p\}} \int_{Q_i} K_i(u_\epsilon)\mathrm{sgn}'_\mu(\phi(u_\epsilon) - \phi(\kappa))\nabla\phi(u_\epsilon)\cdot \mathbf{B}_i \zeta_1\zeta_2 \mathrm{d}x\mathrm{d}t$$
$$+ \sum_{i\in\{h,p\}} \int_{Q_i} K_i(u_\epsilon)\mathrm{sgn}_\mu(\phi(u_\epsilon) - \phi(\kappa))\mathbf{B}_i \cdot \nabla\zeta_2\, \zeta_1 \mathrm{d}x\mathrm{d}t.$$

Let us focus on the first line when $i = h$ (the reasoning when $i = p$ being similar). We consider the flux term

$$I_{\epsilon,\mu} = \int_{Q_h} K_h(u_\epsilon)\mathrm{sgn}'_\mu(\phi(u_\epsilon) - \phi(\kappa))\nabla\phi(u_\epsilon) \cdot \mathbf{B}_h \zeta_1\zeta_2 \mathrm{d}x\mathrm{d}t$$

that has to be carefully studied since we only have weak convergence for $(u_\epsilon)_{\epsilon>0}$ and for $(\nabla\phi(u_\epsilon))_{\epsilon>0}$. That is why we introduce

$$D_\mu(v, w) = \int_w^v (K_h \circ \phi^{-1})(\tau)\mathrm{sgn}'_\mu(\tau - w)\mathrm{d}\tau.$$

So that,

$$\begin{aligned} I_{\epsilon,\mu} &= \int_{Q_h} \nabla(D_\mu(\phi(u_\epsilon), \phi(\kappa))) \cdot \mathbf{B}_h \zeta_1\zeta_2 \mathrm{d}x\mathrm{d}t \\ &= -\int_{Q_h} D_\mu(\phi(u_\epsilon), \phi(\kappa))(\zeta_1\zeta_2 \mathrm{div}\mathbf{B}_h + \zeta_1\nabla\zeta_2 \cdot \mathbf{B}_h)\mathrm{d}x\mathrm{d}t \\ &\quad + \int_{\Sigma_{hp}} D_\mu(\phi(u_\epsilon), \phi(\kappa))\mathbf{B}_h \cdot \nu_h \zeta_1\zeta_2 \mathrm{d}\sigma, \end{aligned}$$

because of Green's formula. We just mention that, since $\phi(u_\epsilon)$ is an element of $L^2(0, T; H^1(\Omega))$, for a.e. t of $]0, T[$, $(\phi(u_\epsilon)_{|\Omega_h})_{|\Gamma_{hp}} = (\phi(u_\epsilon)_{|\Omega_p})_{|\Gamma_{hp}}$. We take now the ϵ-limit through (30). For the boundary integral, $D_\mu(., \phi(k))$ being nonlinear, the weak convergence of the traces of $\phi(u_\epsilon)$ on Σ_{hp} is not sufficient to pass to the limit. That is why we consider the sequence $(D_\mu(\phi(u_\epsilon), \phi(\kappa))\zeta_2)_{\epsilon>0}$. On account of Proposition 4 and since $(D_\mu(., \phi(\kappa)))$ is Lipschitz, $(D_\mu(\phi(u_\epsilon), \phi(\kappa))\zeta_2)_{\epsilon>0}$ strongly converges toward $D_\mu(\phi(u), \phi(\kappa))\zeta_2$ in $L^q(Q_p)$, $1 \le q < +\infty$. Besides, based on a chain-rule argument and estimate (27), we argue that $(D_\mu(\phi(u_\epsilon), \phi(\kappa))\zeta_2)_{\epsilon>0}$ is uniformly bounded in $L^2(0, T; V) \cap L^\infty(Q)$ and so weakly converges (up to a subsequence) toward $D_\mu(\phi(u), \phi(\kappa))\zeta_2$ in $L^2(0, T; V)$. The trace operator from $L^2(0, T; V)$ into $L^2(\Sigma_p)$ being linear and continuous, $(D_\mu(\phi(u_\epsilon), \phi(\kappa))\zeta_2)_{\epsilon>0}$ weakly converges toward $(D_\mu(\phi(u), \phi(\kappa))\zeta_2)$ in $L^2(\Sigma_p)$, and so in $L^2(\Sigma_{hp})$. Then $\lim_{\epsilon\to 0^+} I_{\epsilon,\mu} = I_\mu$, where

$$\begin{aligned} I_\mu &= -\int_{Q_h\times]0,1[} D_\mu(\phi(\pi), \phi(\kappa))(\zeta_1\zeta_2 \mathrm{div}\mathbf{B}_h + \zeta_1\nabla\zeta_2 \cdot \mathbf{B}_h)\mathrm{d}\alpha\mathrm{d}x\mathrm{d}t \\ &\quad + \int_{\Sigma_{hp}} D_\mu(\phi(u), \phi(\kappa))\mathbf{B}_h \cdot \nu_h \zeta_1\zeta_2 \mathrm{d}\sigma. \end{aligned}$$

To take the limit with μ, we come back to the definition of sign'_μ and use the fact that, since $K_h \circ \phi^{-1}$ is continuous on $\phi([M_2(T), M_1(T)])$, $(D_\mu(v, w))_{\mu>0}$ converges toward $\mathrm{sgn}(v - w)K_h \circ \phi^{-1}(w)$ a.e. on $Q_h\times]0, 1[$ and $d\mathcal{H}^n$-a.e. on Σ_{hp}. From the Lebesgue-dominated convergence Theorem, it follows that $\lim_{\mu\to 0^+} I_\mu = I$ where

$$\begin{aligned} I &= -\int_{Q_h\times]0,1[} \mathrm{sgn}(\phi(\pi) - \phi(\kappa))K_h(\kappa)(\zeta_1\zeta_2 \mathrm{div}\mathbf{B}_h + \zeta_1\nabla\zeta_2 \cdot \mathbf{B}_h)\mathrm{d}\alpha\mathrm{d}x\mathrm{d}t \\ &\quad + \int_{\Sigma_{hp}} \mathrm{sgn}(\phi(u) - \phi(\kappa))K_h(\kappa)\mathbf{B}_h \cdot \nu_h \zeta_1\zeta_2 \mathrm{d}\sigma. \end{aligned}$$

Note that $\mathrm{sgn}(\phi(\pi) - \phi(\kappa)) = \mathrm{sgn}(\pi - \kappa)$ a.e. on $Q_h\times]0, 1[$. Eventually,

$$\begin{aligned} &-\int_{Q_p} L_p(u, \kappa, \zeta_1\zeta_2)\mathrm{d}x\mathrm{d}t - \int_{Q_h\times]0,1[} L_h(\pi, \kappa, \zeta_1\zeta_2)\mathrm{d}\alpha\mathrm{d}x\mathrm{d}t \\ &-\int_\Omega |u_0 - \kappa|\zeta_1(0)\zeta_2\mathrm{d}x + \int_{Q_p} \nabla|\phi(u) - \phi(\kappa)| \cdot \zeta_1\nabla\zeta_2\mathrm{d}x\mathrm{d}t \\ &+\int_{\Sigma_{hp}} \{K_h(\kappa)\mathbf{B}_h - K_p(\kappa) \cdot \mathbf{B}_p\} \cdot \nu_h \mathrm{sgn}(\phi(u) - \phi(\kappa))\zeta_1\zeta_2\mathrm{d}\sigma \le 0. \end{aligned} \tag{31}$$

For ζ_2 in $\mathcal{D}(\Omega_h)$, we deduce that

$$-\int_{Q_h\times]0,1[} L_h(\pi,\kappa,\zeta_1\zeta_2)\mathrm{d}\alpha\mathrm{d}x\mathrm{d}t \le \int_{\Omega_h} |u_0-\kappa|\zeta_1(0)\zeta_2\mathrm{d}x.$$

Therefore, by following ideas in [4] or in [5, chap. 2], but here in the context of a process solution, we may be sure that,

$$\operatorname{ess}\lim_{t\to 0^+}\int_{]0,1[\times\Omega_h} |\pi(\alpha,t,x)-\Lambda(x)|\mathrm{d}\alpha\mathrm{d}x \le \int_{\Omega_h}|u_0-\Lambda(x)|\mathrm{d}x, \tag{32}$$

for any bounded measurable Λ on Ω_h and (7) on Ω_h for π is obtained by $\Lambda = u_0$.

Now to establish that (6) is fulfilled, we take in (22) the test function $v = \partial_1 H_l(u_\epsilon,\kappa)\zeta_1\zeta_2$, where ζ_1 belongs to $\mathcal{D}(0,T)$ and ζ_2 to $\mathcal{D}(\overline{\Omega_h})$, $\zeta(t,.)=0$ on Γ_{hp} for any t of $[0,T]$, $\zeta_i \ge 0$ and,

$$\forall l \in \mathbb{N}^\star,\ H_l(z,w) = \left((\operatorname{dist}(z,\mathcal{I}[0,w]))^2 + \left(\frac{1}{l}\right)^2\right)^{1/2} - \frac{1}{l},$$

$$\mathcal{Q}_{h,l}(z,w) = \int_w^z \partial_1 H_l(\tau,w)K_h'(\tau)\mathrm{d}\tau,$$

is the family of boundary entropy–entropy flux pair introduced by Otto [4] (or [5, chap. 2]). We emphasize that $\partial_1 H_l(u_\epsilon,\kappa)\zeta_1\zeta_2$ is an element of $L^2(0,T;H_0^1(\Omega_h))$ so that calculations may be performed as if we were in the single domain Q_h. In particular, the Green formula does not give rise to integrals along the interface. We integrate with respect to the time variable and use the Mignot–Bamberger chain-rule argument. we have:

$$-\int_{Q_h}\left(H_l(u_\epsilon,\kappa)\zeta_2\partial_t\zeta_1 - \mathcal{Q}_{h,l}(u_\epsilon,\kappa)\mathbf{B}_h\cdot\zeta_1\nabla\zeta_2 - \mathcal{G}_{h,l}(u_\epsilon,\kappa)\zeta_1\zeta_2\right)\mathrm{d}x\mathrm{d}t$$
$$\le -\epsilon\int_{Q_h}\partial_1 H_l(u_\epsilon,\kappa)\zeta_1\nabla\zeta_2\cdot\nabla\phi_\epsilon(u_\epsilon)\mathrm{d}x\mathrm{d}t,$$

the convexity of the function $\xi \to H_l(\xi,.)$ being taken into account and

$$\mathcal{G}(u_\epsilon,\kappa) = \int_\kappa^{u_\epsilon}\left(K_h(\tau)\partial_{11}^2 H_l(\tau,\kappa)\right)\mathrm{d}\tau\ \operatorname{div}\mathbf{B}_h + g_h(t,x,u_\epsilon)\partial_1 H_l(u_\epsilon,\kappa).$$

On account of (30) we take the ϵ-limit. It follows:

$$-\int_{]0,1[\times Q_h}\left(H_l(\pi,\kappa)\zeta_2\partial_t\zeta_1 - \mathcal{Q}_{h,l}(\pi,\kappa)\mathbf{B}_h\cdot\zeta_1\nabla\zeta_2 - \mathcal{G}_{h,l}(\pi,\kappa)\zeta_1\zeta_2\right)\mathrm{d}\alpha\mathrm{d}x\mathrm{d}t \le 0.$$

At this point, we adapt Otto's works providing that:

$$\operatorname{ess}\lim_{\tau\to 0^-}\int_{]0,1[\times\Sigma_h\backslash\Sigma_{hp}}\mathcal{Q}_{h,l}(\pi(\alpha,\sigma+\tau\nu),\kappa)\mathbf{B}_h(\bar{\sigma})\cdot\nu_h\zeta\,\mathrm{d}\alpha\mathrm{d}\sigma \le 0,$$

for any ζ of $L_+^1(\Sigma_h\backslash\Sigma_{hp})$. Boundary condition (6) for π follows by observing that $(\mathcal{Q}_{h,l})_{l\in\mathbb{N}^*}$ uniformly converges toward $\mathcal{F}_h(z,0,\kappa)$ as l goes to $+\infty$.

So π "fulfills" (5) with $\zeta \in \mathcal{D}(Q_h)$, (6) and (7) where the integrals over $\Sigma_h\backslash\Sigma_{hp}$, Ω and Q_h are, respectively, turned into integrals over $]0,1[\times\Sigma_h\backslash\Sigma_{hp}$, $]0,1[\times\Omega_h$ and $]0,1[\times Q_h$ with respect to the corresponding measure. This way, by reasoning as in Theorem 1, if $\pi_1(\alpha,.,.)$ and $\pi_2(\beta,.,.)$ are two process solutions for initial data $u_{0,1}$ and $u_{0,2}$, then for a.e. t in $]0,T[$,

$$\int_{]0,1[^2\times\Omega_h}|\pi_1(\alpha,t,x)-\pi_2(\beta,t,x)|\mathrm{d}\alpha\mathrm{d}\beta\mathrm{d}x\mathrm{d}t \le \int_{\Omega_h}|u_{0,1}-u_{0,2}|\mathrm{d}x\,\mathrm{e}^{M'_{g_h}t}.$$

When $u_{0,1} = u_{0,2}$ on Ω_h, there exists a measurable and bounded function u_h on Q_h such that a.e. on Q_h, $u_h(.,.) = \pi_1(\alpha,.,.) = \pi_2(\beta,.,.)$ for a.e. α and β in $]0,1[$. Besides, the uniqueness property ensures that the whole sequence $(u_\epsilon)_{\epsilon>0}$ strongly converges to u_h in $L^q(Q_h)$, $1 \le q < +\infty$. Thus, $u_h = u_{|\Omega_h}$ a.e. on Q_h and from (31) we deduce that u satisfies (5), for any ζ of $\mathcal{D}(0,T) \otimes \mathcal{D}(\Omega)$ so by density for any ζ of $\mathcal{D}(Q)$, and (6). To complete the proof of Theorem 3 we only need to ensure that (7) holds for u. Due to (32) we just have to concentrate on Ω_p. We consider (31) for ζ_2 in $\mathcal{D}(\Omega_p)$:

$$-\int_0^T \left(\int_{\Omega_p} |u-\kappa|\zeta_2 \mathrm{d}x + f(t) \right) \zeta_1'(t)\mathrm{d}t \le \int_{\Omega_p} |u_0 - \kappa|\zeta_2\zeta_1(0)\mathrm{d}x,$$

with

$$f(t) = \int_{\Omega_p}\int_0^t [-\mathrm{sgn}(u(\tau,x)-\kappa)(K_p(u(\tau,x)) - K_p(\kappa))\mathbf{B}_p \cdot \nabla\zeta_2$$
$$+ g_p(\tau,x,u(\tau,x))\mathrm{sgn}(u(\tau,x)-\kappa)\zeta_2 - |\phi(u(\tau,x)) - \phi(\kappa)|\Delta\zeta_2]\mathrm{d}\tau)\mathrm{d}x.$$

So the time-dependent function $t \to \int_{\Omega_p} |u-\kappa|\zeta_2\mathrm{d}x + f(t)$ is identified a.e. with a non-increasing and bounded function, so it has an essential limit when t goes to 0^+, t in $]0,T[\backslash\mathcal{O}$, where $\mathcal{L}(\mathcal{O}) = 0$. As f goes to 0 with t, we obtain

$$\text{ess}\lim_{t\to 0^+} \int_{\Omega_p} |u-\kappa|\zeta_2\mathrm{d}x \le \int_{\Omega_p} |u_0-\kappa|\zeta_2\mathrm{d}x, \tag{33}$$

for any function ζ_2 of $\mathcal{D}(\Omega_p)$, $\zeta_2 \ge 0$, which concludes the proof of Theorem 3. □

5 Conclusion

This paper presents a mathematical analysis of a coupling of hyperbolic/parabolic scalar conservation laws in a bounded domain Ω of $\mathbb{R}^n$, $n \ge 1$. The special framework considered only takes into account the situation of outward characteristics for the first-order operator set in Ω_h; this means that data leave the hyperbolic zone to enter the parabolic one. Hence, from a mathematical point of view, the problem is well-posed in the hyperbolic domain where the existence and uniqueness of a solution is entirely determined by the knowledge of initial and outer-boundary data. To describe the behavior of a solution in the parabolic zone (and to state a uniqueness property), we need to take into account data entering from the hyperbolic domain. In a certain way, the problem may be viewed as uncoupled, but one should not forget that suitable transmission conditions are needed to ensure uniqueness in the parabolic zone. These conditions are not explicitly stated in this paper since they are included in the global formulation itself on the whole domain. They require the continuity of the flux along the interface and are written through an entropy condition between traces coming, respectively, from the parabolic and the hyperbolic area as soon as the latter has a meaning (for example when in the hyperbolic domain, the solution is a bounded function of bounded variation). In fact, as soon as transmission conditions along the interface are available, a uniqueness property for the whole problem defined on the whole studied field is obtained in a forthcoming paper, without a monotonicity assumption on K_h and a sign condition for $\mathbf{B}_h \cdot \nu_h$ along the interface. This way the direction of the characteristics for the first-order operator set in the hyperbolic zone do not need to be a priori determined along the interface between the two domains. For our point of view, if these transmissions cannot be explicitly stated, they need at least to be included in a global formulation on the whole domain as in the present work. This forthcoming work states another existence property that does not refer to a viscous problem. The construction of a solution requires the knowledge of the characteristics field along the interface in order to first define a solution on the parabolic domain and then on the hyperbolic one, or conversely. To release the latter point, a new way to obtain an existence result has to be investigated.

Of course, the previous considerations are meaningless when the operators set on the parabolic and hyperbolic domains are linear. In this situation, no entropy condition is needed to ensure the uniqueness of a solution in the hyperbolic zone; the characteristics are parallel straight lines in the hyperbolic field and so are determined along the interface. The transmission conditions are written as a continuity of the flux and of the trace along the interface, since in this case a weak solution has a trace coming from the hyperbolic area.

References

1. Gastaldi F, Quateroni A (1990) Coupling of two-dimensional hyperbolic and elliptic equations. Comput Methods Appl Mech Eng 80(1–3):347–354
2. Aguilar G, Lisbona F, Madaune-Tort M (1999) Analysis of a nonlinear parabolic-hyperbolic problem. Adv Math Sci Appl 9: 597–620
3. Aguilar G, Lévi L, Madaune-Tort M (2006) Coupling of multidimensional parabolic and hyperbolic equations. J Hyperbolic Differ Eq 3(1):53–80
4. Otto F (1996) Initial-boundary value problem for a scalar conservation law. C R Acad Sci Paris 322 Série I:729–734
5. Malek J, Necas J, Rokyta M, Ruzicka M (1996) Weak and measure-valued solutions to evolutionary PDE's. Chapman & Hall, pp 95–143
6. Peyroutet F, Madaune-Tort M (2001) Error estimate for a splitting method applied to convection–reaction equations. Math Models Appl Sci 11(6):1081–1100
7. Brézis H (1973) Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland Mathematics Studies, 5, Notas de matematica 50. North-Holland Publishing Comp. Amsterdam; American Elsevier Publishing Comp., New York
8. Gagneux G, Madaune-Tort M (1995) Analyse mathématique de modèles non linéaires de l'ingénierie pétrolière. Mathématiques Applications (Paris), 22. Springer-Verlag, 188 pp
9. Eymard R, Gallouët T, Herbin R (1995) Existence and uniqueness of the entropy solution to a nonlinear hyperbolic equation. Chin Ann Math Ser B 16(No.1):1–14
10. Lions JL (1969) Quelques méthodes de résolution des problèmes aux limites non linéaires. Etudes mathématiques. Dunod, Gauthier-Villars, Paris, 554 pp